

# Orbital order driven quantum criticality

ZOHAR NUSSINOV<sup>1</sup> and GERARDO ORTIZ<sup>2</sup>

<sup>1</sup> *Department of Physics, Washington University, St. Louis, MO 63160, USA*

<sup>2</sup> *Department of Physics, Indiana University, Bloomington, IN 47405, USA*

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**Abstract.** - Charge, spin, and orbital degrees of freedom underlie the physics of transition metal compounds. Much work has revealed quantum critical points associated with spin and charge degrees of freedom in many of these systems. Here we illustrate that the simplest models that embody the orbital degrees of freedom - the two- and three-dimensional quantum orbital compass models - exhibit an exact quantum critical behavior on diluted square and cubic lattices (with doping  $\delta = 1/4$  and  $\delta = 1/2$  respectively). This raises the possibility of quantum critical points triggered by the degradation of orbital order upon doping (or applying pressure to) such transition metal systems. We prove the existence of an orbital spin glass in several related systems in which the orbital couplings are made non-uniform. Moreover, a new *orbital Larmor precession* (i.e., a periodic change in the orbital state) is predicted when uniaxial pressure is applied.

*Introduction.* The interplay between charge, spin, and orbital degrees of freedom is a key ingredient underlying the physics of transition metal compounds. The electronic orbital degrees of freedom often allow for cooperative effects leading to well defined spatial orderings. Generally, crystal field effects split the five  $n = 3$  d-wave orbital wavefunctions into a triplet ( $t_{2g}$  orbitals of the  $xy \equiv (Y_{2,-2} - Y_{2,2})/\sqrt{2}$ ,  $yz \equiv (Y_{2,-1} + Y_{2,1})/\sqrt{2}$  or  $xz \equiv (Y_{2,-1} - Y_{2,1})/\sqrt{2}$  types) and an  $e_g$  doublet (spanned by orbitals whose angular dependence is of the  $3z^2 - r^2 \equiv Y_{2,0}$  and  $x^2 - y^2 \equiv (Y_{2,-2} + Y_{2,2})/\sqrt{2}$  forms). Colossal magnetoresistance (CMR) – the sharp decrease of resistivity with magnetic field – occurs in some of the compounds that display orbital orders, e.g. the manganite  $\text{LaMnO}_3$  [1] and its derivatives. In  $\text{LaMnO}_3$ , the orbital order is of a simple character: the state of the single outermost electron in one ion may be that of, say, the  $n = 3$  d-wave state of the  $3z^2 - r^2$  type while it is  $3x^2 - r^2$  on a neighboring ion and so on in a staggered fashion within each plane. Other prominent systems that exhibit various types of orbital order are found among the vanadates (e.g.,  $\text{V}_2\text{O}_3$  [2],  $\text{LiVO}_2$  [3] and  $\text{LaVO}_3$ ) [4] and cuprates (e.g.  $\text{KCuF}_3$ ) [5]. Orbital ordering can be observed via orbital-related magnetism and lattice distortions or by resonant X-ray scattering techniques [6].

Although there are numerous works on quantum criticality in systems with various electronic phases [7, 8]

there is little prior work [9–12] on systems in which the orbital ordering temperature veered to zero. We wish to motivate the following question: *If quantum critical points are associated with the degradation of magnetic/charge/superconducting orders in numerous systems why can these not appear in orbitally ordered systems?* Such quantum critical points are associated with the degradation of orbital order. This path has not been followed despite the success in finding quantum critical points and novel low- $T$  transitions in spin and other electronic systems. We demonstrate the existence of exact quantum criticality driven by orbital order in the simplest of all orbital only models: the  $D = 2$  and 3 dimensional quantum Orbital Compass Models (OCMs). These systems rigorously exhibit order in their classical limit [13–15]. This order is expected to be fortified by quantum effects. Indeed, numerically orbital order was detected in the undiluted  $D = 2$  OCM [11]. We will show that, at a prescribed doping (dilution) of magnitude  $\delta = 1/4$ , this system displays quantum critical correlations. This quantum critical phase may be driven away by applying pressure/strain (i.e. varying the size of the exchange constants corresponding to different spatial directions). Similarly, we show that a particular set of anisotropic couplings of the  $D = 3$  OCM leads to quantum criticality on a lattice of doping  $\delta = 1/2$ . In the particular case that the orbital exchange couplings become random (i.e., non-uniform), we

demonstrate that an *orbital spin glass* may be generated. Furthermore, for any orbital system, when uniaxial pressure is applied the phenomenon of *orbital Larmor precession* develops, i.e., the orbital states change periodically in time as a consequence of the applied pressure.

*Orbital only models.* The orbitals in the transition metal oxides are modeled by  $S = 1/2$   $SU(2)$  degrees of freedom [1, 16]. These pseudo-spins appear in the orbital only Jahn-Teller (JT) interactions and spin-orbital Kugel-Khomskii (KK) Hamiltonians depicting exchange. The simplest Hamiltonian linking interactions with the directionality of the orbitals in space is the OCM [16]. The  $S = 1/2$  anisotropic  $D = 2, 3$  anisotropic OCMs [17] on  $N_{\text{OCM}} = \prod_{\mu} L_{\mu}$  sites of a square or cubic lattices respectively are given, in term of Pauli matrices  $\sigma_j^{\mu}$  with  $\mu = x, y$  ( $D = 2$ ) or  $\mu = x, y, z$  (for  $D = 3$ ), by

$$H_{\text{OCM}} = - \sum_j J_{\mu} \sigma_j^{\mu} \sigma_{j+\hat{e}_{\mu}}^{\mu}. \quad (1)$$

Interactions involving the  $x$ -component of the pseudo-spin  $S_j^x = \sigma_j^x/2$  occur only along the spatial  $x$  direction of the lattice. Similarly, spatial direction-dependent spin-exchange interactions appear for the  $y$  (or  $z$ ) components of the spin. In  $D = 3$ , this model emulates the directional character of the orbital related interactions triggered by JT effects and the orbital only component of the KK Hamiltonian [1, 13, 14, 16] in materials with a single electron in a  $t_{2g}$  level. Varying the strengths of the couplings  $\{J_{\mu}\}$  relative to one another emulates the effects of uniaxial strain and pressures on the orbital-only component of the orbital-dependent spin-exchange (the KK Hamiltonian). Similarly, the influence of uniaxial pressure on the JT interactions may be mimicked by adding a *pseudo-spin magnetic field* linearly coupled to the orbital pseudo-spins. For instance, uniaxial pressure along the cubic  $z$  axis favors orbital states that have little extent along the  $z$  axis. In the case of  $e_g$  orbitals, such a uniaxial pressure favors the planar  $d_{x^2-y^2}$  states (represented by  $\sigma^z = -1$  [1, 13, 14, 16]) vis a vis the cigar-shaped  $d_{3z^2-r^2}$  state (depicted by  $\sigma^z = 1$ ), that is elongated along the  $z$  axis.

Detailed calculations were performed on orbital systems such as the OCM [18]. We will mainly focus on the  $D = 2$  variant of the OCM. By the mappings of [17], the  $D = 2$  OCM is dual to  $(p + ip)$  superconducting Josephson junction arrays on a square lattice. Recent theoretical work suggests similar interactions in  $p$ -band Mott insulators [21].

*The planar orbital compass model.* The  $D = 2$  OCM has the following symmetries [19, 20]  $\hat{O}^{\mu} = \prod_{j \in C_{\mu}} i \sigma_j^{\mu}$ , for  $\mu = x, y$  with  $C_{\mu} \perp \hat{e}_{\mu}$  axis. On a torus, these operators are defined along toric cycles. As  $\hat{O}^{\mu}$  involves  $\mathcal{O}(L^1)$  sites, they constitute  $d = 1$  symmetries [19, 20]. Such symmetries cannot be broken at finite temperatures [20]. These are also symmetries of the diluted model that we will describe below. At the isotropic

point  $J_x = J_y$  it has an additional global reflection symmetry

$$\hat{O}_{\text{reflection}} = \prod_j e^{i \frac{\pi \sqrt{2}}{4} (\sigma_j^x + \sigma_j^y)}, \quad (2)$$

which may be broken at finite temperatures [22].

As we will prove below, the diluted OCM exhibits a quantum critical point at the very same location ( $J_x = J_y$ ) where a finite temperature transition is expected to occur in the undiluted OCM. It is natural to anticipate that the ordering temperature drops monotonically with doping [11] with, as we will show, the system becoming quantum critical for a prescribed dilution of  $\delta = 1/4$  (see Fig. 1). Away from the isotropic point  $J_x = J_y$ , all symmetries are of the  $\hat{O}^{\mu}$  type. At finite temperatures, these symmetries cannot be broken and only topological quantum order may be possible [19, 20]. The algebra satisfied by the bond operators  $A_{ij}^{\mu} = \sigma_i^{\mu} \sigma_j^{\mu}$ , for pairs of sites in which  $\vec{j} - \vec{i} = \hat{e}_{\mu}$ , is particularly simple:  $\{A_{ij}^x, A_{ik}^y\} = \{A_{ik}^y, A_{kl}^x\} = 0$ , ( $j \neq k \neq i$ ),  $[A_{ij}^{\mu}, A_{kl}^{\mu}] = 0$ ,  $(A_{ij}^{\mu})^2 = \mathbb{1}$ ,  $[A_{ij}^{\mu}, A_{kl}^{\mu'}] = 0$ , ( $\mu \neq \mu'$ ,  $i, j \neq k, l$ ). Apart from these algebraic relations, there are no additional constraints that link the bonds  $\{A_{ij}^{\mu}\}$ .

*A doped orbital compass model.* Let us now consider the Hamiltonian of Eq. (1) on the diluted lattice of Fig. 1 (lower panel) with  $N = \frac{3}{4} N_{\text{OCM}}$  sites, and call this new Hamiltonian  $H_{\text{DOCM}}$ . The lattice of Fig. 1 corresponds to a doped (or diluted) system in which  $\delta = 1/4$  of the sites have been replaced by an inert site (dopant). After doping, in addition to the symmetries of  $\hat{O}^{\mu}$  and  $\hat{O}_{\text{reflection}}$ , a new *gauge* symmetry emerges

$$\hat{O}_a = \sigma_a^x, \quad [H_{\text{DOCM}}, \hat{O}_a] = 0, \quad (3)$$

where  $a$  represents sites which are two-fold coordinated, i.e.  $z = 2$  (denoted by squares in Fig. 1). Henceforth we will denote the set of  $z = 2$  sites by  $\Omega_2$ . This gauge symmetry allows to decompose the total Hilbert space into orthogonal subspaces of equal dimensionality  $\mathcal{H} = \bigoplus_{\ell=1}^{2^{N/3}} \mathcal{H}_{\ell}$ , where  $\dim \mathcal{H}_{\ell} = 2^{2N/3}$ . In each sector for all states  $|\phi_n\rangle_{\ell} \in \mathcal{H}_{\ell}$ , we have that  $\sigma_a^x |\phi_n\rangle_{\ell} = \eta_a |\phi_n\rangle_{\ell}$ ,  $\eta_a = \pm 1$ . There are  $2^{N/3}$  sequences of eigenvalues of these operators. Thus, each subspace is labeled by a particular string of  $\pm 1$ , i.e.  $\mathcal{H}_{\ell} = \mathcal{H}_{\{\eta_a\}_{a=1}^{N/3}}$ , with for example  $\{\eta_a\} = +, +, +, -, +, -, \dots, +, +$ . Each of the Hilbert subspaces  $\mathcal{H}_{\ell}$  spans the  $(2N/3)$  spins on the remaining  $2N/3$  ( $z = 3$ ) sites. These sites lie along vertical columns separated by one another by intervening columns of  $z = 2$  sites (see Fig. 1). Let us label the set of all three-fold coordinated sites  $b$  by  $\Omega_3$  (the sets  $\Omega_2$  and  $\Omega_3$  are disjoint  $\Omega_2 \cap \Omega_3 = \{\emptyset\}$ ). The algebra of the bonds  $\{A_{ij}^{\mu}\}$  in each of the projected subspaces  $\mathcal{H}_{\ell}$  is unchanged relative to that defined on the full Hilbert space. To prove this, let us define the projection operators

$$\hat{P}_{\ell} = \prod_{a=1}^{N/3} \left( \frac{\mathbb{1} + \eta_a \sigma_a^x}{2} \right), \quad \hat{P}_{\ell}^2 = \hat{P}_{\ell}, \quad [H_{\text{DOCM}}, \hat{P}_{\ell}] = 0. \quad (4)$$

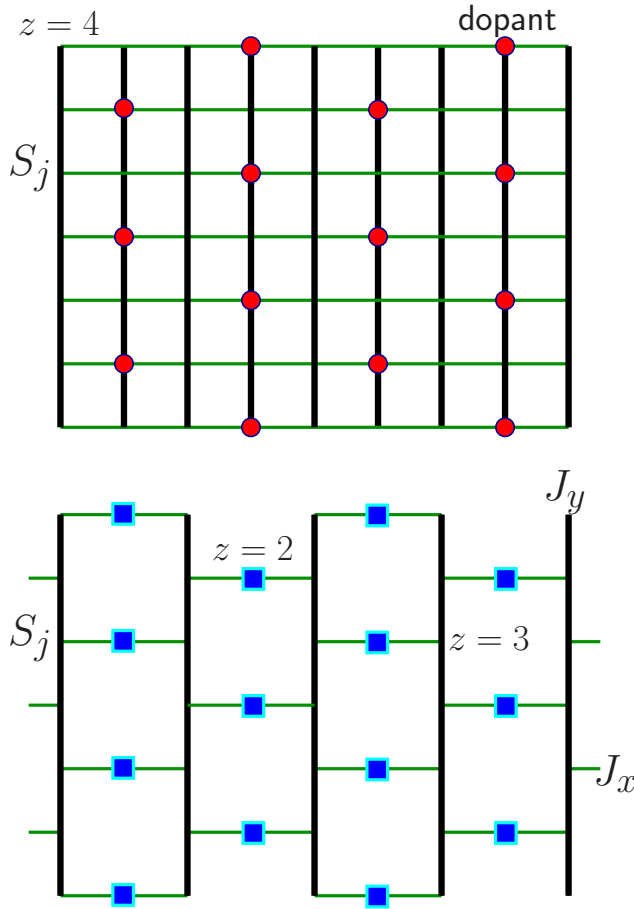


Fig. 1: Doped  $D = 2$  orbital compass model. On each vertex of the graph there is an  $S = 1/2$  degree of freedom indicated by a spin-1/2 operator  $S_j$ . The top panel represents the undoped orbital compass model ( $z = 4$ ) with circles indicating the dopant sites. Vertical links have strength  $J_y$  while horizontal links have strength  $J_x$ . The lower panel displays the resulting doped model where the graph still remains two-dimensional (brick-wall topology) but with two types of vertices: one shares three neighbors ( $z = 3$ ) while the other (squares) is only connected to two neighbors ( $z = 2$ ).

Within each of the projected subspaces, the bond operators  $\bar{A}_{ij}^\mu = \hat{P}_\ell A_{ij}^\mu \hat{P}_\ell$  satisfy relations identical to those above, i.e.  $\{\bar{A}_{ij}^x, \bar{A}_{ik}^y\} = \{\bar{A}_{ik}^y, \bar{A}_{kl}^x\} = 0$ , ( $j \neq k \neq i$ ),  $[\bar{A}_{ij}^\mu, \bar{A}_{kl}^\mu] = 0$ ,  $\hat{P}_\ell (A_{ij}^\mu)^2 \hat{P}_\ell = 1_\ell$ ,  $[\bar{A}_{ij}^\mu, \bar{A}_{kl}^{\mu'}] = 0$ , ( $\mu \neq \mu'$ ,  $i, j \neq k, l$ ). These relations follow directly from the fact that  $[\hat{P}_\ell, A_{ij}^x] = 0$ , since  $[\hat{P}_\ell, \sigma_j^x] = 0$ , and  $[\hat{P}_\ell, A_{ij}^y] = 0$ , because all  $y$ -type bonds  $A_{ij}^y$  have their support entirely in the domain of  $z = 3$  sites  $i, j \in \Omega_3$ . Consequently, the bond algebra does not change.

We now exactly solve the doped OCM (DOCM). The algebraic relations for the interaction terms  $\bar{A}_{ij}^\mu$  defined above are *identical* to those of the interaction terms in the  $D = 1$  transverse field Ising model (TFIM). In both instances, the dimension of the representation of the algebra

is the same as are all additional constraints between the interaction terms. In any given sector  $\mathcal{H}_\ell$  (which amounts to a particular gauge fix), all bonds in the DOCM system are those of decoupled TFIM chains. The Hamiltonian of a transverse field ( $h_j$ ) Ising chain is

$$H_{\text{TFIM}} = - \sum_j \left( h_j \sigma_j^x + J_y \sigma_j^y \sigma_{j+\hat{e}_y}^y \right). \quad (5)$$

On the other hand, the DOCM Hamiltonian of Fig. 1 in the projected subspace,  $\bar{H}_{\text{DOCM}} \equiv \hat{P}_\ell H_{\text{DOCM}} \hat{P}_\ell$ , is

$$\bar{H}_{\text{DOCM}} = - \sum_b \left( J_x \eta_a \sigma_b^x + J_y \sigma_b^y \sigma_{b+\hat{e}_y}^y \right), \quad (6)$$

where  $a$  denotes the  $z = 2$  site  $a \in \Omega_2$  that is a nearest-neighbor of a given  $z = 3$  site  $b \in \Omega_3$  ( $a$  lies to the immediate left or right of  $b$  in Fig. 1).  $\bar{H}_{\text{DOCM}}$  represents the Hamiltonian of a set of  $D = 1$  TFIM chains of length  $L$  correlated by the gauge fields  $\eta_a$ , but otherwise uncoupled. This is equivalent to say that  $\bar{H}_{\text{DOCM}}$  represents a *single*  $D = 1$  TFIM chain of length  $2N/3$  and transverse fields  $h_b = J_x \eta_a$ . The spectrum of a TFIM chain is independent of the sign of  $h_j$ . Given the unitary (and hermitian) operator  $U_\Gamma = \prod_{j \in \Gamma} \sigma_j^y$ , where  $\Gamma$  denotes an arbitrary set of lattice sites,  $U_\Gamma H_{\text{TFIM}}[\{h_j\}] U_\Gamma = H_{\text{TFIM}}[\{\bar{h}_j\}]$ , where  $\bar{h}_j = -h_j$  if  $j \in \Gamma$ , and  $\bar{h}_j = h_j$ , otherwise. Thus, all different projected  $\bar{H}_{\text{DOCM}}$  have exactly the same spectrum (although their eigenstates are interchanged). As each sector  $\mathcal{H}_\ell$  leads to an identical partition function, the resulting one for the DOCM is

$$\mathcal{Z} = \text{tr}_{\mathcal{H}} e^{-\beta H_{\text{DOCM}}} = \sum_\ell \text{tr}_{\mathcal{H}_\ell} e^{-\beta \bar{H}_{\text{DOCM}}} = 2^{N/3} \mathcal{Z}_{\text{TFIM}}, \quad (7)$$

where  $\mathcal{Z}_{\text{TFIM}}$  is the partition function of the TFIM of Eq. (5) with  $2N/3$  sites. If the system is on a torus of size  $L_x \times L_y$  we can impose periodic boundary conditions if the dimensions of the lattice are such that there is an even number of sites along a vertical column and a number of sites that is a multiple of four along the horizontal direction, i.e.  $L_x \equiv 0 \pmod{4}$ ,  $L_y \equiv 0 \pmod{2}$ . In each sector  $\mathcal{H}_\ell$  in addition to the algebraic relations defined above, the vertical bonds will satisfy the additional constraint:  $\prod_{b=1}^{L_y} \bar{A}_{b,b+\hat{e}_y} = 1$ . Analogously, the TFIM of Eq. (5) in which the bonds satisfy the analogous constraint  $\prod_{b=1}^{L_y} \sigma_b^y \sigma_{b+\hat{e}_y}^y = 1$  corresponds to a system of decoupled chains (formed by the  $z = 3$  sites) in each of which there is a periodic boundary condition (the chain is of length  $L_y$ ). For a system of decoupled  $L_x/2$  TFIM chains the partition function is  $\mathcal{Z}_{\text{TFIM}} = (z_{\text{TFIM}})^{L_x/2}$ , where  $z_{\text{TFIM}}(L_y)$  is the partition function of a TFIM chain of length  $L_y$ .  $H_{\text{TFIM}}$  can be diagonalized by a Jordan-Wigner transformation followed by a Bogoliubov transformation,  $H_{\text{TFIM}} = \sum_k \epsilon_k (\gamma_k^\dagger \gamma_k - 1/2)$ , with  $\gamma_k$  ( $\gamma_k^\dagger$ ) the fermionic annihilation (creation) operator for a fermion (Bogoliubov quasiparticle) of wavenumber  $k$  (the set of allowed wavenumbers

is  $\{-\pi, -\pi + \pi/L_y, \dots, 0, \dots, \pi - \pi/L_y\}$ . The quasiparticle energies of a TFIM chain of length  $L_y$  are given by  $\epsilon_k = 2\sqrt{J_x^2 + J_y^2 - 2J_x J_y \cos k}$ , and its ground state energy  $E_0 = -\sum_k \epsilon_k/2$ . Thus, the grand potential per site is

$$\frac{\Omega}{N} = -\frac{2E_0}{3L_y} - \frac{k_B T \ln 2}{3} - \frac{2k_B T}{3L_y} \sum_k \ln[1 + z e^{-\beta \epsilon_k}], \quad (8)$$

with  $z = \exp[\beta \xi]$  the fugacity and  $\xi$  the chemical potential. Thus, the DOCM of Fig. 1 has a quantum critical point at  $J_x = \pm J_y$  ( $D = 2$  Ising universality class). Using our mapping, in the DOCM, the correlator

$$G(\vec{r}, t) = \langle \sigma^y(\vec{r}, t) \sigma^y(0, 0) \rangle_{\mathcal{H}} = \langle \sigma^y(\vec{r}, t) \sigma^y(0, 0) \rangle_{\text{TFIM}}. \quad (9)$$

Due to the decoupling to TFIM chains, the last average is zero unless  $\vec{r}$  lies along the  $y$  axis. With  $c = 2J_y$  (the lattice constant is set to one), for  $T = 0$ , we have at the quantum critical point  $G(\vec{r}, 0) \sim (|y|/c)^{-1/4} \delta_{x,0}$ , and is manifestly critical. At the quantum critical point for general  $T > 0$  temperatures, the Fourier transformed correlator is given by [7]

$$\tilde{G}(k, \omega) \propto \frac{\Gamma(\frac{1}{16} - i\frac{\omega + ck}{4\pi k_B T}) \Gamma(\frac{1}{16} - i\frac{\omega - ck}{4\pi k_B T})}{\Gamma(\frac{15}{16} - i\frac{\omega + ck}{4\pi k_B T}) \Gamma(\frac{15}{16} - i\frac{\omega - ck}{4\pi k_B T})}. \quad (10)$$

By our mapping, expressions are similarly available in the paramagnetic and ordered phases. The regular pattern of doped sites leads to the emergence of the gauge symmetry  $\hat{O}_a$ , crucial for the exact solvability of the model. However, essential to observe lower dimensional ( $d = 1$ ) physics is the presence of the  $d = 1$   $\hat{O}^\mu$  symmetries [19]. A modest amount of randomness in the doping (i.e. not with a regular pattern) does not invalidate the dimensional reduction argument, although we can no longer solve the problem exactly. This conclusion is supported by the numerical simulations of Ref. [11].

Recently, non-uniform structures that may be generated by doping orbital systems were investigated in Ref. [23].

*The doped  $D = 3$  orbital compass model.* Consider a cubic lattice in which consecutively layered planes (stacked along the cubic  $z$  direction) have the following form:  $ABCBABCBA \dots$ . Here,  $B$  planes are as in Fig. 1. Planes of the  $A$  and  $C$  types are those in which the corresponding doped columns of plane  $B$  are fully doped. Furthermore, along the columns that are undoped in plane  $B$ , planes  $A$  and  $C$  are half-doped with every other site removed such that there is a relative shift, between the  $A$  and  $C$  planes, of one lattice constant along the  $y$  direction. The average dopant density on this lattice is  $\delta = 1/2$ . Consider the  $D = 3$  OCM of Eq. (1) on this lattice. The symmetry of Eq. (3) remains intact on all planes of the  $B$  type, while  $\{\sigma_u^z\}$  for all undoped sites  $u$  in the  $A$  and  $C$  planes are local symmetries. Replicating the solution of the  $D = 2$  case, we find that the  $D = 3$  DOCM reduces to decoupled TFIM chains in the

$B$  planes as before with a transverse field of strength  $h = \sqrt{J_x^2 + J_z^2}$ . This is so as along the undoped planar columns of Fig. 1, each site now feels two transverse fields along the  $x$  and  $z$  directions. The additional interactions along the  $z$  direction originate from a site in the  $A$  or  $C$  planes. No additional  $x$  or  $y$  interactions appear within the  $A$  or  $C$  planes. The dispersion is now given by  $\epsilon_k = 2\sqrt{J_x^2 + J_y^2 + J_z^2 - 2J_y \sqrt{J_x^2 + J_z^2} \cos k}$ . This particular lattice exhibits quantum critical points along the locus  $J_x^2 + J_z^2 = J_y^2$ . As a function of *uniaxial pressure* along the  $y$  axis, the point  $J_y = J\sqrt{2}$  (with  $J_x = J_z \equiv J$ ) is quantum critical. For increasing/decreasing uniaxial pressures, and their influence on the orbital-dependent spin-exchange,  $G(\vec{r}, t)$  exhibits the correlations of the ordered/paramagnetic phases of the TFIM [24].

*Uniaxial pressure on Jahn-Teller orbital-only interactions: Orbital Larmor precession.* Thus far, we examined the (diluted) anisotropic OCM. The latter captures the effects of uniaxial strain and pressure as these apply to the orbital component of the orbital-dependent spin-exchange interactions. We now turn to the influence of the uniaxial strain/pressure on the direct (JT borne) orbital interactions. Following our earlier discussion, for a uniaxial pressure/strain along the  $\nu$  space direction, these now generally lead to a Hamiltonian of the form

$$H_{\text{OCM}; \text{ strain}} = H_{\text{OCM}} - H_P, \quad (11)$$

where the uniaxial pressure is represented by

$$H_P = \sum_j P_\nu \sigma_j^\nu, \quad (12)$$

with no summation over  $\nu$ . For the spatial directions  $\nu = x$  or  $y$  (here one needs to perform a unitary transformation), the Hamiltonian of Eq. (11) can, once again, be exactly solved in the two- and three-dimensional dilutions considered hitherto. Let us consider the case  $\nu = x$ . In this case, the symmetries of Eq. (3) remain manifestly unchanged. Following our earlier derivation, we arrive at a simple generalization of Eq. (6)

$$\bar{H}_{\text{DOCM}; \text{ strain}} = \bar{H}_{\text{DOCM}} - \sum_b P_x \sigma_b^x - \sum_a P_x \eta_a, \quad (13)$$

i.e., a transverse field Ising model with an effective transverse field  $h_b = J_x \eta_a + P_x$  while, as before, the Ising exchange constant  $J = J_y$ . Unlike Eq. (5), (13) does not exhibit a symmetry of the spectrum under local inversions of the type  $\eta_a \rightarrow -\eta_a$ . The general case of uniaxial strain cannot be exactly solved and leads to a richer system. We have seen that the pressure induced orbital-dependent spin-exchange interactions may, on their own, lead to a uniaxial pressure tuned quantum critical point. We cannot trivially solve for the thermodynamics once the direct orbital JT interactions of the form of Eqs. (12) and (6) are included. The crux is the trace over all  $\{\eta_a\}$  that should



be performed which in effect sums over the partition function of all subsystems with different transverse fields  $h_b$ . Hamiltonian (13) can be exactly solved within each  $\{\eta_a\}$  sector. It may well be that the quantum critical behavior persists or that, e.g., a Griffiths-type phase appears in place [7].

We now comment on a related feature of the diluted OCM when spatially non-uniform couplings are present. Consider a general system in which the (random) exchange constants  $\{J_{\mu,j}\}$  vary from bond to bond (defined by  $(j, j + \hat{e}_\mu)$ ). Replicating our derivation above, we find that after applying a uniform external field that emulates the effect of uniaxial strain/pressure, we arrive at a transverse field random Ising model (TFRIM) within each sector of  $\{\eta_a\}$ ,

$$H_{\text{TFRIM}} = - \sum_b \left( [J_{x,b}\eta_a + P_x] \sigma_b^x + J_{y,b} \sigma_b^y \sigma_{b+\hat{e}_y}^y \right). \quad (14)$$

In Eq. (14),  $P_x$  plays the role of a uniform transverse field while  $\{J_{y,j}\}$  are random longitudinal exchange constants. As the TFRIM exhibits spin glass behavior with well known characteristics, so may does our system. We may thus predict the appearance of an *orbital spin glass* [26].

A fundamental consequence of the form of the uniaxial pressure/strain induced JT interaction is the appearance of *orbital Larmor precession*. This precession happens even in the simplest bare orbital system. If only the consequences of uniaxial pressures along a direction  $\nu$  are considered, then for *any* orbital system, the Hamiltonian  $H_P$  of Eq. (12) leads to orbital Larmor precessions. That is, the orbital states of the electrons change periodically in time due to the application of pressure

$$\frac{d\vec{\sigma}}{dt} = \gamma \vec{\sigma} \times \vec{P}, \quad (15)$$

with  $\vec{P} = \sum_\nu P_\nu \hat{e}_\nu$ , and  $\gamma$  the effective gyromagnetic ratio of the orbital system. The precession times scale as  $|\vec{P}|^{-1}$ . In principle, our new orbital precessions should be observable.

*Possible realizations of orbital ordered triggered quantum criticality.* Our work shows that in orbital systems, upon the application of external pressure (uniaxial strain) and varying doping, we may attain quantum criticality associated with the degradation of orbital order. What physical systems may display new orbital fluctuation driven quantum critical points? The Luttinger liquid, a  $D = 1$  electronic liquid, and some variants thereof are prototypical examples of critical systems. (Indeed, our exact solution relied on a *dimensional reduction* to decoupled chains ensured by the symmetries of these models.) In some physically probed orbital systems, e.g. [25], an effective reduction in dimensionality to precisely such a system occurs due to the arrangement of orbitals.  $\text{KCuF}_3$  has a weakly coupled spin chain structure and offers an example of a system with a  $D = 1$  (Luttinger liquid) like behavior observed over a broad range of temperatures, momenta, and frequencies [27]. The orbital only component

of the orbital dependent spin-exchange shares the same features and anisotropic form as the orbital only interactions [1, 5]. The directional character of the orbitals in  $\text{KCuF}_3$  triggers a quasi one-dimensional behavior of the spins. Ionic substitution of  $\text{KCuF}_3$  (to  $\text{KCu}_{1-\delta}\text{Zn}_\delta\text{F}_3$ ) [9] revealed a vanishing of the orbital ordering temperature for a doping of  $\delta = 1/2$ . A reduction in the orbital ordering temperature occurs as Zn does not have an orbital degree of freedom. References [10,11] noted that this doping fraction is smaller than the one needed to eradicate order in typical diluted magnets (e.g.  $\text{KMn}_{1-\delta}\text{Mg}_\delta\text{F}_3$ ) [28,29]; in typical magnetic systems, the decrease in the ordering temperature and its saturation are governed by the percolation threshold (where the ordering temperature vanishes at the critical dopant concentration of  $\delta_c = 0.69$  for the simple cubic lattice). The faster degradation of orbital order with doping vis a vis percolation physics can indeed be attributed [11] to the directional character of the orbital exchange interactions. A loss of orbital order was also seen in the bilayered ruthenate  $\text{Ca}_3\text{Ru}_2\text{O}_7$  [12] with a suggested critical pressure of  $P \simeq 55$  kbar; this degradation of orbital order correlates with a metal to insulator transition. Our work demonstrates the possibility that *quantum criticality is associated with the vanishing orbital ordering temperature* of systems such as  $\text{KCu}_{1-\delta}\text{Zn}_\delta\text{F}_3$  and may be accessed, and its signatures probed for, by varying parameters such as the external pressure and doping. Orbital-only interactions (either borne directly by the orbital-only Jahn-Teller effect or by the orbital component of the orbital-dependent spin-exchange) are pertinent to numerous other systems. These include, for example, the layered cuprates ( $\text{K}_2\text{CuF}_4$ ), and manganites ( $\text{La}_2\text{MnO}_4$  or  $\text{La}_2\text{Mn}_2\text{O}_7$ ). Such systems were discussed recently in [23].

Note added in proof: Several months after the initial appearance of our work, Ref. [23] discussed non-uniform structures that may appear in doped orbital systems described by orbital models.

## REFERENCES

- [1] VAN DEN BRINK J., KHALIULLIN G., and KHOMSKII D., *Colossal Magnetoresistive Manganites*, edited by CHATTERJI T. (Kluwer Academic Publishers, Dordrecht) 2002, **and references therein**.
- [2] CASTELLANI C., NATOLI C. R., and RANNINGER J., *Phys. Rev. B*, **18** (1978) 4945.
- [3] PEN H. F., VAN DEN BRINK J., KHOMSKII D. I., and SAWATZKY G. A., *Phys. Rev. Lett.*, **78** (1978) 1323.
- [4] KHALIULLIN G., HORSCH P., and OLES A. M., *Phys. Rev. Lett.*, **86** (2001) 3879.
- [5] KUGEL K. I. and KHOMSKII D. I., *Sov. Phys. (JETP)*, **37** (1973) 725; *Sov. Phys. Usp.*, **25** (1982) 231.
- [6] MURAKAMI Y. *et al.*, *Phys. Rev. Lett.*, **80** (1998) 1932; MURAKAMI Y. *et al.*, *Phys. Rev. Lett.*, **81** (1998) 582; ENDOH Y. *et al.*, *Phys. Rev. Lett.*, **82** (1999) 4328; ISHIHARA S. and MAEKAWA S., *Phys. Rev. Lett.*, **80** (1998) 3799; ELFI-MOV I. S., ANISIMOV V. I., and SAWATZKY G. A., *Phys. Rev. Lett.*, **82** (1999) 4264.

- [7] SACHDEV S., *Quantum Phase Transitions* (Cambridge University Press, Cambridge) 2000.
- [8] VARMA C. M., NUSSINOV Z., and VAN SAARLOOS W., *Physics Reports*, **361** (2002) (5-6), and references therein.
- [9] TATAMI N., NIIOKA S., and MURAKAMI Y., (unpublished).
- [10] TANAKA T., MATSUMOTO M., and ISHIHARA S., *Phys. Rev. Lett.*, **95** (2005) 267204.
- [11] TANAKA T. and ISHIHARA S., *Phys. Rev. Lett.*, **98** (2007) 256402.
- [12] J. F. KARPUS *et al.*, *Phys. Rev. B*, **73** (2006) 134407.
- [13] NUSSINOV Z., BISKUP M., CHAYES L., and VAN DEN BRINK J., *Europhysics Letters*, **67** (2004) 990.
- [14] BISKUP M., CHAYES L., and NUSSINOV Z., *Comm. Math. Phys.*, **255** (2005) 253.
- [15] MISHRA A. *et al.*, *Phys. Rev. Lett.*, **93** (2004) 207201.
- [16] VAN DEN BRINK J., *New J. of Phys.*, **6** (2004) 201, and references therein.
- [17] NUSSINOV Z. and FRADKIN E., *Phys. Rev. B*, **71** (2005) 195120.
- [18] LIECHTENSTEIN A. I., ANISIMOV V. I., and ZAAENEN J., *Phys. Rev. B*, **52** (1995) R5467; MIZOKAWA T. and FUJIMORI A., *Phys. Rev. B*, **54** (1996) 5368; ISHIHARA S., INOUE J., AND MAEKAWA S., *Phys. Rev. B*, **55** (1997) 8280; KHALIULLIN G. and OUDOVENKO V., *Phys. Rev. B*, **56** (1997) R243; FEINER L. F. and OLES A. M., *Phys. Rev. B*, **59** (1999) 3295; VAN DEN BRINK J., *Phys. Rev. Lett.*, **87** (2001) 217202.
- [19] NUSSINOV Z. and ORTIZ G., [cond-mat/0605316](#), [cond-mat/0702377](#); *Phys. Rev. B*, **77** (2008) 064302.
- [20] BATISTA C. D. and NUSSINOV Z., *Phys. Rev. B*, **72** (2005) 045137.
- [21] ZHAO E. and LIU W. V., [arXiv: 0801.0589](#); WU C., [arXiv:0801.0888](#).
- [22] In addition to Eq. (2), the operator

$$U = \exp[\sum_j (i\pi/4)\sigma_j^z]$$

also effects such an exchange with the difference that  $J_x \rightarrow J_y$  while  $J_y \rightarrow (-J_x)$ .

- [23] KUGEL K. I. *et al.*, [arXiv:0805.4806](#).
- [24] Obviously, for  $\delta = 1/2$  we can also have every other site removed (all sites in which the sum of the  $x$ ,  $y$ , and  $z$  coordinates is even). That doping corresponds to a set of decoupled Ising spins. Similarly, if all sites with an even  $x$  coordinate are removed, decoupled  $D = 2$  OCMs appear (in planes parallel to the  $yz$  plane).
- [25] LEE S. *et al.*, *Nature Materials*, **5** (2006) 471.
- [26] FISHER D. S. and HUSE D. A., *Phys. Rev. Lett.*, **56** (1986) 1601; *Phys. Rev. B*, **38** (1988) 386; SCHECHTER M. and LAFLORENCIE N., *Phys. Rev. Lett.*, **97** (2006) 137204.
- [27] LAKE B., TENNANT D. A., FROST C. D., and NAGLER S. E., *Nature Materials*, **4** (2005) 329.
- [28] STINHCORBE R. B., *Phase Transition and Critical Phenomena*, edited by DOMB C. and LEBOWITZ J. L., Vol. **7** (Academic Press, London) 1983.
- [29] BREED D. J., GILJAMSE J. W., STERKENBURG J. W. E., and MIEDEMA A. R., *J. Appl. Phys.*, **41** (1970) 1267.